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A modified Newton-type method with sixth-order convergence for solving nonlinear equations

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Abstract

In this paper, we present and analyze a sixth-order convergent iterative method for solving nonlinear equations. The method is free from second derivatives and permits f'(x) =0 in iteration points. Some numerical examples illustrate that the presented method is more efficient and performs better than classical Newton's method.

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Keywords: Newton-type method; Nonlinear equations; Order of convergence; Iterative method; Efficiency index

1. Introduction

In this paper, we consider iterative methods to find a simple root α of a nonlinear equation

$$f(x)=0, \tag{1}$$

where $f: D\subseteq R\rightarrow R$ for an open interval D is a scalar function and it is sufficiently smooth in a neighborhood of α.

It is well known that classical Newton's method (NM for simplicity) is a basic and important method for solving non-linear equation by the iterative scheme

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},$$

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which is quadratically convergent in the neighborhood of α. In recent years, much attention has been given to develop iterative methods for solving nonlinear equations and many iterative methods have been developed [1-11].

Motivated and inspired by the on going activities in this direction, in this paper, we present a sixth-order convergent method for solving nonlinear equations. Several examples are given to illustrate the efficiency of the iterative method.

2. The New Method and Its Convergence

Let us consider the following iterative method.

Algorithm 1. For given x_0, we consider the iteration method for solving non-linear equation by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{\alpha_n f(x_n) + f'(x_n)},$$

(2)

$$z_{n+1} = y_n - \frac{f(y_n)}{\beta_n f(y_n) + f'(y_n)},$$

(3)

$$x_{n+1} = z_n - \frac{f(z_n)}{\gamma_n f(z_n) + f'(y_n)},$$

(4)

where α_n, β_n, γ_n are real numbers chosen such that

$$0 \le |\alpha_n|, |\beta_n|, |\gamma_n| \le 1,$$

and

$$\mathrm{sign}(\alpha_n f(x_n)) = \mathrm{sign}(f'(x_n)),$$

$$\mathrm{sign}(\beta_n f(y_n)) = \mathrm{sign}(f'(y_n)),$$

$$\mathrm{sign}(\gamma_n f(z_n)) = \mathrm{sign}(f'(y_n)),$$

where $n=1,2,\ldots$, sign(x) is the sign function.

For Algorithm 1, we have the following convergence result.

Theorem 1. Assume that the function $f: D \subseteq R \rightarrow R$ has a single root α, where D is an open interval. If f(x) has first, second and third derivatives in the interval D, then Algorithm 1 is sixth-order convergent in a neighborhood of α and it satisfies error equation

$$e_{n+1} = 2c_2(\alpha_n + c_2)^3(\beta_n + c_2)e_n^6 + O(e_n^7), \tag{5}$$

where

$$e_n = x_n - \alpha,\ c_k = \frac{f^{(k)}(\alpha)}{k!\,f'(\alpha)}, k=2,3,\ldots.$$

Proof. Let α be the simple root of f(x), $c_k=\frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$, k=2,3,… and $e_n=x_n-\alpha$. Consider the iteration function F(x) defined by

$$F(x)=\frac{f(z(x))}{\gamma_n f(z(x))+f'(y(x))}, \tag{6}$$

where

$$y(x)=x-\frac{f(x)}{\alpha_n f(x)+f'(x)},$$

$$z(x)=y(x)-\frac{f(y(x))}{\beta_n f(y(x))+f'(y(x))}.$$

By some computations using Maple we can obtain

$$F(\alpha)=\alpha,$$
$$F^{(i)}(\alpha)=0,\ i=1,2,3,4,5,$$
$$F^{(6)}(\alpha)=\frac{45f''(\alpha)[f''(\alpha)+2\alpha_n f'(\alpha)]^3[f''(\alpha)+2\beta_n f'(\alpha)]}{f'(\alpha)^5}. \tag{7}$$

Furthermore, from the Taylor expansion of $F(x_n)$ around α, we get

$$\begin{aligned} x_{n+1}&=F(x_n)\\ &=F(\alpha)+F'(\alpha)(x_n-\alpha)+\frac{F''(\alpha)}{2!}(x_n-\alpha)^2+\frac{F'''(\alpha)}{3!}(x_n-\alpha)^3+\frac{F^{(4)}(\alpha)}{4!}(x_n-\alpha)^4\\ &\quad+\frac{F^{(5)}(\alpha)}{5!}(x_n-\alpha)^5+\frac{F^{(6)}(\alpha)}{6!}(x_n-\alpha)^6+O((x_n-\alpha)^7). \end{aligned} \tag{8}$$

Substituting (7) into (8) yields

$$x_{n+1}=\alpha+e_{n+1}=2c_2(\alpha_n+c_2)^3(\beta_n+c_2)e_n^6+O(e_n^7). \tag{9}$$

Therefore, we have

$$e_{n+1}=2c_2(\alpha_n+c_2)^3(\beta_n+c_2)e_n^6+O(e_n^7), \tag{10}$$

which means the order of convergence of Algorithm 1 is six.

The proof is completed.

Now, we consider efficiency index defined as $p^{1/w}$, where p is the order of the method and w is the number of function evaluations per iteration required by the algorithm. It is not hard to see that the efficiency index of the Algorithm 1 is 1.43097 which is better than that of classical Newton's method 1.4142.

3. Numerical Experiments

Now, we employ Algorithm 1 to solve some nonlinear equations and compare it with NM. Displayed in Table 1 are the number of iterations (ITs) required such that $|f(x_n)|<10^{-14}$.

In table 1, we use the following functions.

$f_1(x) = \sin(x+1)-x+2$, $\alpha=2.07076672709785$.

$f_2(x) = x^3+e^x-x+1$, $\alpha=-1.38070588484698$.

$f_3(x) = x^3-10$, $\alpha=2.15443469411846$.

$f_4(x) = x^3-2x^2+x-1$, $\alpha=1.75491057842537$.

$f_5(x) = (x-1)^3-1$, $\alpha=2$.

Table 1. Comparison of Algorithm 1and NM

Functions	x_0	NM	Algorithm 1
f_1	-1	Failure	4
f_1	0	65	2
f_2	0	Failure	5
f_2	-1.2	7	2
f_3	0	Failure	6
f_3	0.01	22	5
f_4	1	Failure	5
f_4	1.02	10	6
f_5	1	Failure	7
f_5	1.05	28	8

The computational results in Table 1 show that Algorithm 1 requires less ITs than NM. Therefore, the present sixth-order convergent method is of practical interest and can compete with NM.

4. Conclusions

We construct a new sixth-order convergent Newton-type method for solving nonlinear equations. The new algorithm is free from second derivatives and permits $f'(x)=0$ in some iterates. Analysis of efficiency shows that the new method is more efficient and performs better than classical Newton's method particularly in the case when the derivatives of the function $f(x)$ at some iterates are singular or almost singular.

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